Int J Theor Phys (2010) 49: 1212–1221
DOI 10.1007/s10773-010-0301-x

Entanglement Dynamics and Transfer for Two-Qubit Werner-Like States in Markovian Environments

Xiu-Xing Zhang · Guang-Hou Sun · Xun-Chang Yin · Chuan-Jia Shan

Received: 2 January 2010 / Accepted: 1 March 2010 / Published online: 19 March 2010

Abstract We investigate entanglement dynamics and transfer in a system of two identical independent qubits, each of them locally interacting with a bosonic reservoir. Starting from two-qubit extended Werner-like states, we have shown that the degree of entanglement of the initial states, Markovian environments and the purity can control the time of the two-qubit entanglement sudden death and the reservoirs' entanglement sudden birth. Moreover, the phenomenon of entanglement sudden death/birth may occur depending on the values of parameters like purity or degree of entanglement of the initial state. When initial states are not pure, entanglement sudden death/birth always occurs, this will permit us to link the occurrence time of entanglement sudden death/birth and entanglement transfer to the purity or the degree of entanglement of the initial states.

Keywords Entanglement dynamics · Werner-like states · Markovian environments

1 Introduction

Entanglement plays a central role in the application of in quantum computing and quantum communication [1–4]. Great effort has been devoted to studying bipartite entanglement dynamics in different quantum systems, such as cavity QED [5, 6], trapped ions [7], spin systems [8, 9], atomic ensembles [10] and photon pairs [11]. However, due to coupling

X.-X. Zhang
Department of Physics and Electronic Engineering, Weinan Teachers University, Weinan 714000, China

G.-H. Sun
School of Science, Jiujiang University, Jiujiang 332005, China

X.-C. Yin
School of Physics and Electric Engineering, Anqing Teachers College, Anqing 246011, China

C.-J. Shan (✉)
College of Physics and Electronic Science, Hubei Normal University, Huangshi 435002, China
e-mail: scj1122@163.com

to noisy environments that eventually leads to the fast decay of entanglement, it has been shown that two qubits which do not interact directly with each other but interact with environment can be entangled or disentangled. The phenomenon named as entanglement sudden death exhibits striking difference with the usual local decoherence, which has been theoretically predicted by Yu and Eberly [12], and experimentally observed for entangled photon pairs [13] and atomic ensembles [14]. The decoherence process is indeed one of the major obstacles in quantum information processing. So the study of the controlled entanglement dynamics is of much importance to the prospects of maintaining quantum information. Entanglement dynamics and decoherence have been studied in the frame of various models [15–19]. Typically, entanglement sudden death occurs when the two qubits interact with two independent environments. A completely different phenomenon appears when the qubits interact with their independent environments, respectively. Recently, a deeper understanding of the sudden death process has been gained by considering the quantum correlations shared by the environments which show a sudden birth [20]. In addition, due to their fundamental importance in quantum information processing and quantum computation, quantum dissipative systems have attracted much attention in recent years. Some studies have also showed that entanglement of qubits will revive in the case of a commonly shared reservoir [21–24] or of independent reservoirs [25–27].

Due to its great importance, here we are concerned with in detail the study of entanglement dynamics in Markovian approximation for two qubits interacting, respectively, with two independent reservoirs. Different from the previous work, the mixed extended Werner-like state, including pure and mixed states is taken into account. The influence of entanglement of the initial states and the purity of the initial state on the ESD is also studied. Our results show that a transfer of atomic entanglement into two reservoirs is possible in the Markovian environments. When initial states are pure, it is interesting to point out that the entanglement sudden death/birth phenomenon disappears by controlling the entanglement of the initial states. We also find the ESD of two atoms may appear in the evolution and the ESD time can be retarded by increasing the purity of the initial state of the two atoms. Furthermore, under the conditions of different purity and initial entanglement, sudden death of atomic entanglement is accompanied by entanglement sudden birth of reservoirs, and vice versa. Our results will be helpful in understanding the origin of the ESD/ESB and the relation between the ESD/ESB and different mixed portions in the initial states.

2 The Theoretical Hamiltonian Model

Now, we consider a system consisting of two qubits each coupled to a zero-temperature bosonic reservoir in the vacuum denoted, respectively. In this paper, we will assume that each atom-reservoir system is isolated and the reservoirs are initially in the vacuum state while the atoms are initially in extended Werner-like state.

Under the rotating-wave approximation, the Hamiltonian of the interaction between an atom and an N-mode reservoir is

$$\hat{H} = \omega\hat{\sigma}_+\hat{\sigma}_- + \sum_{j=1}^{N}\omega_j\hat{b}_j^\dagger\hat{b}_j + \sum_{j=1}^{N}g_j(\hat{\sigma}_-\hat{b}_j^\dagger + \hat{\sigma}_+\hat{b}_j), \tag{1}$$

where $\hat{b}_j^\dagger$, $\hat{b}_j$ are the creation and annihilation operators of the mode j of the reservoir, $\hat{\sigma}_+ = |1\rangle\langle 0|$, $\hat{\sigma}_- = |0\rangle\langle 1|$ and ω are the inversion operators and transition frequency of the qubit; ω_j and g_j are the frequency of the mode j of the reservoir and its coupling strength with the qubit.

For an initial state of the form $|1\rangle \otimes |\bar{0}\rangle_r$ with $|\bar{0}\rangle_r = \prod_{j=1}^{N} |0_j\rangle_r$, then the time evolution of the single qubit-reservoir system is given by

$$|\Phi(t)\rangle = C_0(t)|1\rangle|\bar{0}\rangle_r + \sum_{j=1}^{N} C_j(t)|0\rangle|1_j\rangle_r, \tag{2}$$

where $|1_j\rangle_r$ is the state of the reservoir with only one exciton in the jth mode. Setting $\delta_j = \omega - \omega_j = 0$, the equations for the probability amplitudes take the form

$$\dot{C}_0(t) = -i \sum_{j=1}^{N} g_j C_j(t), \tag{3}$$

$$\dot{C}_j(t) = -i g_j^* C_0(t). \tag{4}$$

Formally integrating (4) and inserting its solution into (3), we obtain a integro-differential equation for $C_0(t)$,

$$\dot{C}_0(t) = -\sum_{j=1}^{M} |g_j|^2 \int_0^t dt_1 C_0(t_1). \tag{5}$$

In the continuum limit for the reservoir spectrum the sum over the modes is replaced by the integral $\sum_{j=1}^{M} |g_j|^2 \to \int d\omega J(\omega)$, where $J(\omega)$ is the reservoir spectral density. In the following we focus on the case in which the structured reservoir is the electromagnetic field inside a lossy cavity. In this case, the atom is interacting resonantly with the reservoir with Lorentzian spectral density

$$J(\omega) = \frac{R^2}{\pi} \frac{\lambda}{(\omega - \omega_c)^2 + \lambda^2}, \tag{6}$$

where the weight R is proportional to the vacuum Rabi frequency and λ is the width of the distribution and therefore describes the pseudomode decay rate into the reservoir, and ω_c is the fundamental frequency of the cavity. Typically, according to [28], weak-coupling ($\lambda > 2R$), where the behavior of the qubit-reservoire system is Markovian and irreversible decay occurs, and strong-coupling regime ($\lambda < 2R$), where non-Markovian dynamics occurs accompanied by an oscillatory reversible decay and a structured rather than a flat reservoir situation applies. In our paper, we will mainly make considerations to the Markovian approximation.

Through introducing the correlation function $f(t - t_1) = \int d\omega J(\omega) e^{i(\omega_c - \omega)(t - t_1)}$ and performing the Laplace transform of (5), we acquire

$$s\tilde{C}_0(s) - C_0(0) = -\tilde{C}_0(s)\tilde{f}(s). \tag{7}$$

From the above equation we can derive the quantity $\tilde{C}_0(s)$. Finally, inverting the Laplace transform, we obtain a formal solution for the amplitude

$$C_0(t) = e^{-\lambda t/2}\left[\cosh(\Omega t/2) + \frac{\lambda}{\Omega}\sinh(\Omega t/2)\right], \quad \lambda > 2R \tag{8}$$

where $\Omega = \sqrt{|\lambda^2 - 4R^2|}$. If we set $C(t) = \sqrt{1 - C_0(t)^2}$, (2) can be rewritten as

$$|\Phi(t)\rangle = C_0(t)|1\rangle|\bar{0}\rangle_r + C(t)|0\rangle|\bar{1}\rangle_r, \tag{9}$$

with $|\bar{1}\rangle_r = \frac{1}{C(t)} \sum_{j=1}^{M} C_j(t)|1_j\rangle_r$.

Now, we study the joint evolution of two identical two-qubit systems. For the initial state of qubit-pair AB, instead of Bell-like and Werner states [29], we shall consider the following extended Werner-like state

$$\rho^I_{AB}(0) = r|\phi\rangle_{ABAB}\langle\phi| + \frac{1-r}{4} I_{AB}, \tag{10}$$

with r the purity of the initial state of qubits AB, I_{AB} the 4×4 identity matrix and

$$|\phi\rangle_{AB} = (\cos\theta|00\rangle + \sin\theta|11\rangle)_{AB}, \tag{11}$$

the Bell-like state. Obviously, the state in (11) reduces to the standard Werner state when $\theta = \pi/4$ and to Bell-like pure state when $r = 1$. The atom and its reservoir now evolve as an effective two-qubit system. We will study the joint evolution of atoms A and B with their corresponding reservoirs a and b initially in the global state

$$|\rho^I(0)\rangle_{ABab} = |\rho^I\rangle_{AB} \otimes |\bar{0}\bar{0}\rangle_{ab}, \tag{12}$$

respectively. The time evolution of the composite system reads

$$|\rho^I(t)\rangle_{ABab} = U(t)|\rho^I\rangle_{AB} \otimes |\bar{0}\bar{0}\rangle_{ab} U^\dagger(t), \tag{13}$$

where the coefficients for the states $|\rho^I(t)\rangle_{ABab}$ are given by (8).

3 Entanglement Dynamics

In order to describe the entanglement dynamics of the bipartite system, we use the concurrence, which is proposed by Wootters [29] as a measure of entanglement. The concurrence $C = 0$ corresponds to a separable state and $C = 1$ to a maximally entangled state. For a system described by the above density matrix, which can denote either a pure or a mixed state, the concurrence is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{14}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues in a decreasing order of the spin-flipped density operator R defined by $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, $\tilde{\rho}$ denotes the complex conjugate of ρ, σ_y is the usual Pauli matrix.

It is noted that the extended Werner-like states belong to the class of the X states whose density matrix is of the form

$$\rho = \begin{pmatrix} x & 0 & 0 & v \\ 0 & y & u & 0 \\ 0 & u^* & z & 0 \\ v^* & 0 & 0 & w \end{pmatrix} \tag{15}$$

with x, y, z, w real positive and u, v complex quantities. For the X-state equation (15), the concurrence can be derived as

$$C(\rho) = 2\max\{0, |u| - \sqrt{xw}, |v| - \sqrt{yz}\}. \tag{16}$$

Firstly, the total system state at $t = 0$ is $|\rho^I(0)\rangle_{ABab}$. After time t, the reduced density matrix $|\rho^I(t)\rangle_{ABab}$ can be obtained by tracing over the degrees of freedom of qubits a and b, which remains the X-form with

$$\begin{aligned}
x &= \frac{1-r}{4} + r(\cos^2\theta + \sin^2\theta|\chi_A|^2|\chi_B|^2),\\
y &= \frac{1-r}{4} + r\sin^2\theta|\chi_A|^2|\xi_B|^2,\\
z &= \frac{1-r}{4} + r\sin^2\theta|\xi_A|^2|\chi_B|^2,\\
w &= \frac{1-r}{4} + r\sin^2\theta|\xi_A|^2|\xi_B|^2),\\
u &= 0, \qquad v = r\sin\theta\cos\theta\xi_A^*\xi_B^*.
\end{aligned} \tag{17}$$

And the reduced density matrix form of $\rho_{ab}^{\phi}(t)$ is the same as $\rho_{AB}^{\phi}(t)$ but with the following matrix elements

$$\begin{aligned}
x &= \frac{1-r}{4} + r(\cos^2\theta + \sin^2\theta|\xi_A|^2|\xi_B|^2),\\
y &= \frac{1-r}{4} + r\sin^2\theta|\xi_A|^2|\chi_B|^2,\\
z &= \frac{1-r}{4} + r\sin^2\theta|\chi_A|^2|\xi_B|^2,\\
w &= \frac{1-r}{4} + r\sin^2\theta|\chi_A|^2|\chi_B|^2,\\
u &= 0, \qquad v = r\sin\theta\cos\theta\chi_A^*\chi_B^*.
\end{aligned} \tag{18}$$

The functions $\xi_j(t)$, $\chi_j(t)$ are given by

$$\xi_j(t) = e^{-\lambda_j t/2}\left[\cosh(\Omega_j t/2) + \frac{\lambda_j}{\Omega_j}\sinh(\Omega_j t/2)\right], \tag{19}$$

$$\chi_j(t) = \sqrt{1 - \xi_j(t)^2}, \tag{20}$$

where $j = A, B$. For simplicity, $\lambda_A = \lambda_B = \lambda$, $\Omega_A = \Omega_B = \Omega$. By virtue of equation (16), we can get the concurrences $C(\rho_{AB}^{\phi}(t))$ and $C(\rho_{ab}^{\phi}(t))$ as

$$C_{AB(ab)}^{\phi} = 2\max\{0, |v| - \sqrt{yz}\}. \tag{21}$$

In this section we will analyze the entanglement evolution in the double atom-reservoir model. The two-qubit entanglement dynamics has been previously analyzed taking pure Bell-like states as initial states. However, the extended Werner-like(EWL) states have the following advantages. First, we can easily find that the density matrix at arbitrary time t is still X structure. Second, the EWL states allow us to clearly show the influence of the purity and the amount of entanglement of the initial states on the entanglement dynamics simultaneously. So it appears of interest to study the entanglement dynamics of two independent qubits, each locally interacting with a reservoir, in the case of more general initial conditions and in particular for Werner states. The procedure we have developed is applicable to any two-qubit initial state and therefore also to extended Werner-like states. The dynamics of bipartite entanglement between the qubits of our model show interesting features.

Figures 1 and 2 show the concurrences between the two atoms ($C(\rho^{\phi}_{AB}(t))$) and the two reservoirs ($C(\rho^{\phi}_{ab}(t))$) as a function of the time t and various values of the parameter θ with pure Bell-like states as initial states ($r = 1$) in the Markovian regime ($\lambda = 5R$). For various values of the parameter θ, that is to say, the two qubits are initially in different entangled states, we find that for some initial states, the entanglement between the two atoms occurs sudden death, and sudden birth of entanglement between the two reservoirs arises.

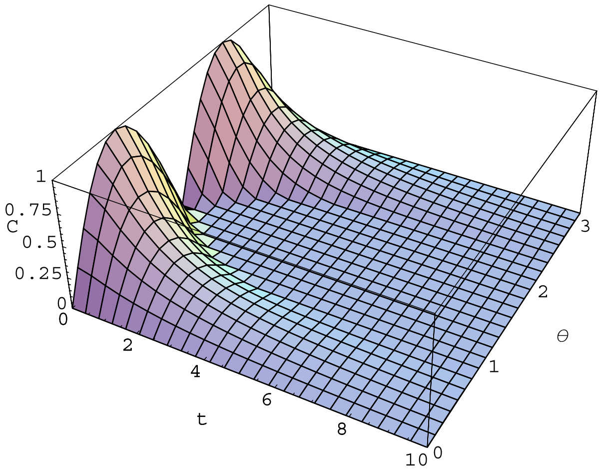


Fig. 1 Time evolution of concurrences between the two atoms ($C(\rho^{\phi}_{AB}(t))$) for various values of the parameter θ with pure Bell-like states as initial states ($r = 1$) in the Markovian regime ($\lambda = 5R$)

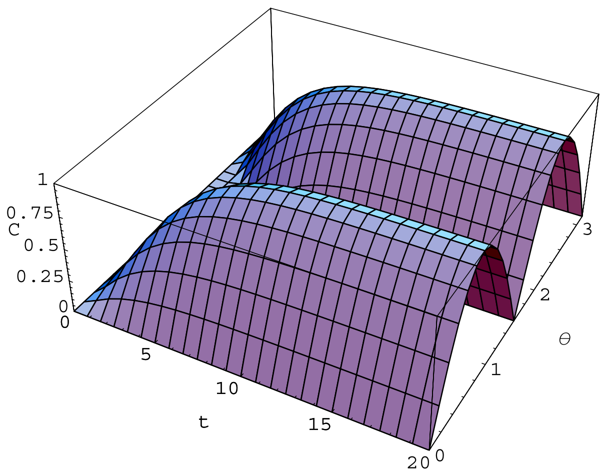


Fig. 2 Time evolution of concurrences between the two reservoirs ($C(\rho^{\phi}_{ab}(t))$) for various values of the parameter θ with pure Bell-like states as initial states ($r = 1$) in the Markovian regime ($\lambda = 5R$)

According to irreversible decay effect of the Markovian reservoirs, two atoms entanglement disappears and does not revive after their entanglement sudden death, and the reservoirs can only present entanglement sudden birth phenomenon; while under some initial conditions, the atoms entanglement can not disappear after a finite time without entanglement sudden death of two atoms, and the reservoirs' entanglement sudden birth does not occur. That is to say, we can prepare certain initial entanglement states with no ESD or ESB phenomena to prolong entanglement time.

The purity of the EWL states is dependent on the parameter r and the amount of the entanglement of the EWL states are related to r and θ. In this part, we will focus on the entanglement dynamical evolution of the composite system in extended Werner-like states, i.e., for $r \neq 1$. The choice of θ corresponds to the Bell-like initial states, while the choice of the purity rests on the fact that in this condition the initial state for the atoms is more realistic. In Figs. 3 and 4, we show the concurrence as a function of both t and the dimensionless parameter r. The results show that the ESD of two atoms and ESB of two cavities occur in some situations but at different times, the differences depend on the values

Fig. 3 Evolution of two-qubit concurrences C_{AB} as a function the dimensionless parameter r with $\theta = \pi/4$ and $\lambda = 5R$

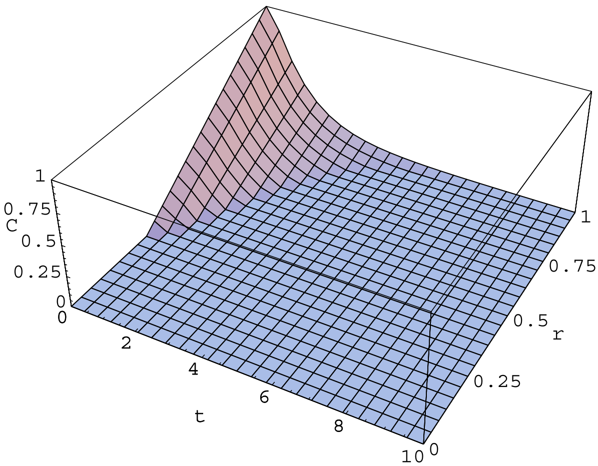
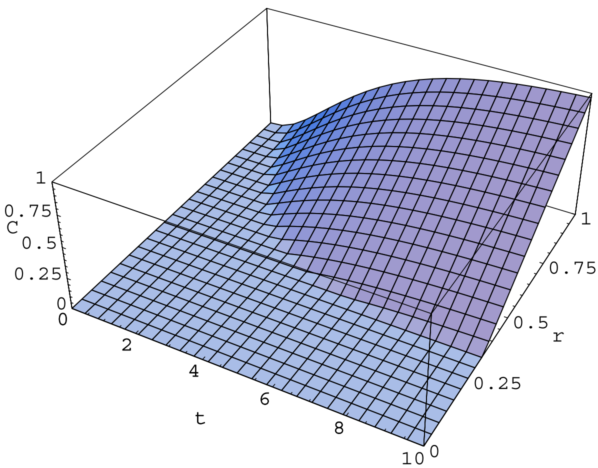


Fig. 4 Evolution of two-reservoir concurrences C_{ab} as a function the dimensionless parameter r with $\theta = \pi/4$ and $\lambda = 5R$



of purity r. In addition, one can also observe that there are two interesting features in the entanglement evolution. One is that the pairwise entanglement oscillates at the high purity, but as the impurity increases the entanglement evolves to zero and will remain zero without the entanglement revivals. A physical interpretation of the result is that when the impurity is very low, the effects of impurity are not strong enough to weaken the entanglement, so continuous entanglement evolution appears. But the higher impurity the stronger the effects of impurity on the entanglement evolution, as a result, entanglement sudden death and birth become more obvious with increasing impurity. The other is that for intermediate values of purity r, entanglement revivals after finite dark periods occur for the initial Werner-like states. When $r < 1/3$, the entanglement of two atoms and two reservoirs will remain zero, which can be seen in Figs. 3 and 4. That is to say, entanglement and entanglement transfer exist for these states when $r > 1/3$.

Now we will turn our attention to the effect of the values of λ on the entanglement degree. When we consider the case that the different pseudomode decay λ is in each lossy cavity, such as $\lambda = 3, 4, 6R$ (also satisfy the condition of Markovian effects). From Fig. 5, the atomic entanglement can disappear eventually, and the entanglement between the two reservoirs can reach a stationary value after a certain time which is equal to the initial entanglement of the two atoms. This indicates that the information initially stored in the atoms is completely transferred to the reservoirs, irrespective of the Markovian effect. We can see that entanglement sudden death and sudden birth can appear much more slowly with the

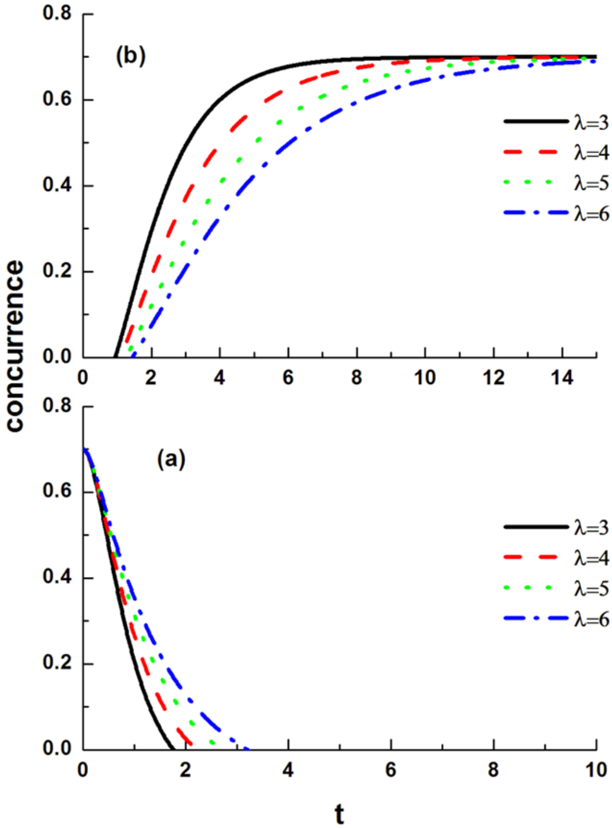


Fig. 5 Evolution of two-qubit concurrences C_{AB} (**a**) and C_{ab} (**b**), for different parameter λ with $\theta = \pi/4$, $r = 0.8$

increase of λ. As we show, concurrence actually goes abruptly to zero in a finite time and remains zero thereafter. That is to say, the entanglement sudden death always happens in the Markovian reservoirs. Meanwhile, the entanglement sudden birth of two reservoirs always appears. We also find that the speeds of disentanglement (loss of entanglement) and decoherence depend on pseudomode decay λ. From the figures, the lower the value of pseudomode decay λ, the longer time two atoms remain entanglement sudden death, While the larger the value of pseudomode decay λ, the longer time two reservoirs remain entanglement sudden birth. When the atomic entanglement disappears thoroughly, it is clearly shown that the reservoirs entanglement arrives to the value of the two atoms' initial entanglement. That is because the atomic entanglement is also transferred to the Ab, Ba, Aa and Bb subsystems during the time evolution of the two-qubit entanglement, and when these entanglements disappear completely, the reservoirs entanglement would reach the value of the two atoms' initial entanglement. The initial entanglement of atoms shall be transferred into the reservoirs in Markovian environments. That is to say, in the lossy cavities system, it is interesting to find that the initial atomic entanglement with Werner-like states can be totally transferred to the two reservoirs system.

4 Conclusions

In summary, we investigate the sudden birth and sudden death of entanglement of two qubits which are prepared in two-qubit extended Werner-like states and interact with uncorrelated structured reservoirs. Under the conditions of Markovian environments lossy cavities and the purity, the atomic entanglement is transferred to the reservoirs' entanglement thoroughly, the larger pseudomode decay can make this entanglement transfer more easily. We also find that Markovian environments and the purity can control the time of the two-qubit entanglement sudden death and the reservoirs' entanglement sudden birth. We believe that our results contribute in shedding light on the behavior of quantum entanglement in realistic conditions, that is when the effects of the Markovian environment and the purity on the quantum system are taken into account. The experimental implementation seems to be feasible due to the recent advances in deterministic trapping of atoms in the optical cavities [30, 31]. In physical contexts, the observation of the effects we have discussed should be achievable with the current experimental technologies.

Acknowledgements This work is supported by the National Natural Science Foundation of China under Grant No. 10904033, Natural Science Foundation of Hubei Province Grant No. 2009CDA145, Educational Commission of Hubei Province under Grant No. D20092204 and the Postgraduate Programme of Hubei Normal University under Grant No. 2007D20.

References

1. Bennett, C.H., et al.: Phys. Rev. Lett. **70**, 1895 (1993)
2. Wei, H., Deng, Z.J., Zhang, X.L., Feng, M.: Phys. Rev. A **76**, 054304 (2007)
3. Grover, L.: Phys. Rev. Lett. **80**, 4329 (1998)
4. Shan, C.J., Liu, J.B., Liu, T.K., Huang, Y.X., Li, H.: Int. J. Theor. Phys. **48**, 1516 (2009)
5. Zheng, S.B., Guo, G.C.: Phys. Rev. Lett. **85**, 2392 (2000)
6. Shan, C.J., Xia, Y.J.: Acta Phys. Sin. **55**, 1585 (2006)
7. Turchette, Q.A., Wood, C.S., King, B.E., Myatt, C.J., Leibfried, D., Itano, W.M., Monroe, C., Wineland, D.J.: Phys. Rev. Lett. **81**, 3631 (1998)
8. Wang, X.G.: Phys. Rev. A **64**, 012313 (2001)
9. Wang, X.G.: Phys. Rev. A **66**, 044305 (2002)

10. Julsgaard, B., Kozhekin, A., Polzik, E.S.: Nature **413**, 400 (2001)
11. Aspect, A., Grangier, P., Roger, G.: Phys. Rev. Lett. **47**, 460 (1981)
12. Yu, T., Eberly, J.H.: Phys. Rev. Lett. **93**, 140404 (2004)
13. Aolita, L., Chares, R., Cavalcanti, D., Acín, A., Davidovich, L.: Phys. Rev. Lett. **100**, 080501 (2008)
14. Almeida, M.P., de Melo, F., Hor-Meyll, M., Salles, A., Walborn, S.P., Souto Ribeiro, P.H., Davidovich, L.: Sience **316**, 579 (2007)
15. Man, Z.X., Xia, Y.J., An, N.B.: J. Phys. B **41**, 085503 (2008)
16. Yu, T., Eberly, J.H.: Phys. Rev. Lett. **97**, 140403 (2006)
17. Jing, J., Lü, Z.G., Yang, G.H.: Phys. Rev. A **76**, 032322 (2007)
18. Zhang, Y.J., Man, Z.X., Xia, Y.J.: J. Phys. B **42**, 095503 (2009)
19. Chen, Z.Y., Zhang, G.F.: Opt. Commun. **275**, 274 (2007)
20. López, C.E., Romero, G., Lastra, F., Solano, E., Retamal, J.C.: Phys. Rev. Lett. **101**, 080503 (2008)
21. Yönaç, M., Yu, T., Eberly, J.H.: J. Phys. B **40**, S45 (2007)
22. Ficek, Z., Tanas, R.: Phys. Rev. A **77**, 054301 (2008)
23. Maniscalco, S., Francia, F., Zaffino, R.L., Gullo, N.L., Plastina, F.: Phys. Rev. Lett. **100**, 090503 (2008)
24. Li, Y., Zhou, J., Guo, H.: Phys. Rev. A **79**, 012309 (2009)
25. Bellomo, B., Lo, Franco R., Compagno, G.: Phys. Rev. A **77**, 032342 (2008)
26. Dajka, J., Mierzejewski, M., Lczka, J.: Phys. Rev. A **77**, 042316 (2008)
27. Cao, X.F., Zheng, H.: Phys. Rev. A **77**, 022320 (2008)
28. Dalton, B.J., Barnett, S.M., Garraway, B.M.: Phys. Rev. A **64**, 053801 (2001)
29. Wooters, W.K.: Phys. Rev. Lett. **80**, 2245 (1998)
30. Boozer, A.D., Boca, A., Miller, R., Northup, T.E., Kimble, H.J.: Phys. Rev. Lett. **97**, 083602 (2006)
31. Fortier, K.M., Kim, S.Y., Gibbons, M.J., Ahmadi, P., Chapman, M.S.: Phys.Rev. Lett **98**, 233601 (2007)